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The effect of a small background inhomogeneity on the asymptotic properties of linear perturbations☆

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ABSTRACT

General regularities in the evolution of one-dimensional unstable linear perturbations on a weakly inhomogeneous background are studied when, at the initial instant, the perturbations are concentrated in the δ-neighbourhood of a certain point. Times are considered when these perturbations do not fall outside the limits of a certain domain of size l such that $\delta \ll l \ll L$, where L is the large characteristic size of the background inhomogeneity. With contain assumptions, the effect of the background inhomogeneity on the asymptotic behaviour of the perturbations at long times is taken into account in a general form. The first corrections to the well known asymptotic relation for the evolution of perturbations on a homogeneous background, that arise because of background inhomogeneity, are obtained using Hamilton's method. An example of the use of the proposed approximate method is considered and the error in the approximation is estimated.

1. Introduction. Hamilton's method

The methods employed below in studying the asymptotic behaviour of linear perturbations at long times on a weakly inhomogeneous background are close to the well-known methods used in the case of a homogeneous background,[1,2] and we shall therefore briefly mention the results relating to this case which are required later.

The homogeneous linear equations describing the propagation of perturbations on a homogeneous background admit of solutions, the dependence of which on the Cartesian coordinate X and the time τ is represented in the form $\exp[i(kX-\omega\tau)$, where the wave number k and the frequency ω are related by a dispersion equation $D(\omega, k)=0$ which ensures the existence of a non-trivial solution of the above mentioned type.

If perturbations with initial data, concentrated in a certain finite interval, are considered, then, using an expansion of the initial data in a Fourier integral, the solution can be represented in the form

$$u(X,\tau) = \int_{-\infty}^{\infty} f(k) e^{i(kX-\omega(k)\tau)} dk, \quad \tau \geq 0 \tag{1.1}$$

Here, integration is carried out over real values of k, and $f(k)$ is the Fourier transform of the initial data, which are assumed to be given for $\tau=0$ in a sufficiently small neighbourhood of the point $X=0$. The relation $\omega(k)$ is obtained from the dispersion equation. If $\omega(k)$ is a multi-valued function of k, expressions (1.1) should be replaced by the sum of similar integrals over all branches of the function $\omega(k)$. The quantities f and u can denote vectors (that is, the sets f_i and u_i, $(i=1, 2, \ldots, n)$). The asymptotic form of the perturbations (1.1) on the rays $X=\upsilon\tau$ for long τ is found by the method of steepest descent,[3] where the saddle point should satisfy the equation

$$d\omega(k)/dk = \upsilon, \quad \upsilon = X/\tau \tag{1.2}$$

Equality (1.2) enables us to find the stationary points $k=k_\upsilon$ of the exponent in integral (1.1). The most difficult problem in determining the asymptotic forms of integral (1.1) lies in determining which of the stationary points is the saddle point, determining the asymptotic

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forms of integral (1.1). The quantity $\mathrm{Im}(\omega - k\upsilon)]$ at the saddle point characterizes the rate of growth or attenuation of the perturbation $u(X, \tau)$ on a ray $X = \upsilon\tau$ as the time τ increases.

In the case when the background on which the perturbations propagate depends slowly on X, that is, it depends on X/L, where L is assumed to be a large quantity, it is natural to replace the exponent in expression (1.1) by

$$i\left[\int_0^X k(X', \omega)dX' - \omega\tau\right] \equiv i\Theta(X, \tau, \omega) \tag{1.3}$$

Here, $k(X, \omega)$ is the function k obtained from the dispersion equation $D(\omega, k, X) = 0$, involving a slow dependence on X. It is assumed that $D(\omega, k, X)$ is an analytic function of its arguments and that a domain of real values of k and X exists in which $\mathrm{Im}\,(\omega) > 0$ and, moreover, $\mathrm{Im}(\omega)$ is not considered to be a small quantity (it does not vanish when $L \to \infty$). If a segment of length $\Delta X \gg L$ and a certain short time interval $\Delta\tau$ are considered, then a small increment in expression (1.3) is represented in the form $i(k\Delta X - \omega\Delta\tau)$, that is, in a form corresponding to the index of the exponent in equality (1.1), which enables us to conclude that

$$\partial\Theta/\partial X = k, \quad \partial\Theta/\partial\tau = -\omega$$

In the case of an inhomogeneous background, equality (1.1) has to be replaced by

$$u = \int_{-\infty}^{\infty} f(k_0)e^{i\Theta,\,(X, \tau, \omega)}dk_0 \tag{1.4}$$

Here, the quantity ω is considered as a function of the variable k_0 when $X = 0$:

$$\omega(k_0) = \omega(k_0, 0) \tag{1.5}$$

For given sufficiently large X and τ, both the values of the function $\Theta(X, \tau, \omega)$ as well as of the exponent in equality (1.1) are assumed to be large and rapidly varying functions of ω (or k_0) and integral (1.4) can therefore be approximately evaluated by the method of steepest descent. The saddle point should satisfy the equation $\partial\Theta/\partial k_0 = 0$ or (assuming that $d\omega/dk_0 \neq \infty$) the equation $\partial\Theta/\partial\omega = 0$, which gives

$$\int_0^X \frac{\partial k(\omega, X')}{\partial\omega}dX' = \tau \tag{1.6}$$

Note that the integrand is the inverse of the group velocity, equality (1.6) expresses the fact that a perturbation propagating over the complex plane X, having emerged from the point $X = 0$ at $\tau = 0$ and moving at a group velocity $\partial\omega/\partial k$, arrives at a point X after a time τ.

When X and τ are given, this equality enables us to find the corresponding value of ω, the value of k_0 according to equality (1.5) and, also, to evaluate integral (1.4) approximately, if the stationary point found is the saddle point. If ω is fixed and X and τ are assumed to be variable, Eq. (1.6) enables us to obtain the trajectory $X(\tau)$ of the perturbation corresponding to this value of ω. To do this, it is sufficient to differentiate equality (1.6) with respect to τ, which leads to the system of equations

$$\frac{d\tau}{dX} = \frac{\partial k(X, \omega)}{\partial\omega}, \quad \omega = \mathrm{const} \tag{1.7}$$

that is equivalent to Hamilton's system

$$\frac{dX}{d\tau} = \frac{\partial\omega}{\partial k}, \quad \frac{dk}{d\tau} = -\frac{\partial\omega}{\partial X} \tag{1.8}$$

with Hamiltonian $\omega(k, X)$. It is a characteristic system of the Hamilton–Jacobi equation representing the dispersion equation $\omega = \omega(k, X)$ after ω and k have been replaced by $-\partial\Theta/\partial\tau$ and $\partial\Theta/\partial X$:

$$\frac{\partial\Theta}{\partial\tau} + \omega\left(\frac{\partial\Theta}{\partial X}, X\right) = 0 \tag{1.9}$$

The Hamiltonian system of equations is widely used to study the behaviour of perturbations on a weakly inhomogeneous background.[1,4,5] A number of general results has been obtained in the case when $\mathrm{Im}(\omega(k, X)) \ll \mathrm{Re}\,(\omega(k, X))$.[1,6,7,8,9] As in the case when a perturbation evolves on a homogeneous background, the saddle point should be determined among the stationary points of the function $\Theta(X, \tau, \omega)$. It is well known that the Hamilton system of equations has a first integral $\omega(k, X)$ which takes constant values for the solution $X(\tau)$, $k(\tau)$ of the Hamilton equations, and that ensures the equivalence of systems of equations (1.7) and (1.8).

Differentiating equality (1.3) with respect to τ when $\omega = \mathrm{const}$, we obtain

$$\frac{d\Theta}{d\tau} = k\frac{dX}{d\tau} - \omega \tag{1.10}$$

This equation, together with Eqs. (1.7) or (1.8), serves to find the solution of Hamilton–Jacobi equation (1.9).

The solutions of Hamilton's equations on a weakly varying background are studied below, and the function Θ is calculated approximately. It is assumed that, for given X and τ, the stationary point of the function $\Theta(X, \tau, \omega(k_0))$, which is considered as the function of k_0, remains close to the corresponding saddle point of integral (1.1) and, consequently, this stationary point will be the saddle point determining the asymptotic forms of integral (1.4).

Note that the representation of the solution in the form of (1.1), where the function $\Theta(X, \tau)$ satisfies Hamilton–Jacobi equation (1.9), is approximate and holds for small background inhomogeneities. If the approach considered here is compared with the WKB method, it can be concluded that it takes account of the exponential dependence of the solution on time and the coordinate and ignores the finite pre-exponential factors. The latter is admissible when the characteristic time and space scale are large, such that $\mathrm{Im}(\Theta)$ is a large quantity and $|\exp(i\Theta)|$ determines the order of magnitude of the solution.

The behaviour of the function $\Theta(X, \tau)$ under various additional assumptions is considered next.

2. Approximate description of perturbations on a weakly varying background

Perturbations on a weakly varying background will be considered. Thus, if the characteristic length, over which the background changes considerably, is equal to L, it is assumed in the problem considered below that the initial data, when $\tau = 0$, are non zero in the interval δ, and times are considered when the perturbations do not fall outside the limits of a domain, the size of which is l. Moreover, it is assumed that

$$\delta \ll l \ll L \tag{2.1}$$

It can be said that the asymptotic form of the perturbations for $\varepsilon \to 0$ when

$$l = \varepsilon^{-1}\delta, \quad L = \varepsilon^{-1} l$$

is considered.

This supposition enables us to use the asymptotic representation of the perturbations, assuming that it remains close to the corresponding representation of the perturbations on a homogeneous background. Hence, the first corrections in the asymptotic evolution of initially localized perturbations, which appear due to background inhomogeneity, will be found in general form.

A similar problem has been considered[10] for a specific form of the equations, characterized by the fact that the function $\omega(k, X)$ is represented as a sum of two terms, one of which depends solely on k and the other depends solely on X. In the general case, finding the solution reduced to constructing quite complicated series in X and τ. However, it is of interest to take into account the terms in the dispersion equation involving k and X simultaneously since, as it has been shown,[11] these terms may have a decisive effect on the behaviour of perturbations by stopping their growth. The use of Hamilton's method has a number of advantages and, primarily the advantages of simplicity and generality.

In satisfying conditions (2.1), it is convenient instead of X and τ, to introduce the dimensionless variables

$$x = X/L, \quad t = \tau/T \tag{2.2}$$

Everywhere in this paper it is implied that the asymptotic forms are considered when $L \to \infty$, $T \to \infty$ and the dispersion equation is fixed. We will also assume that k and ω do not change in the case of this transformation. According to relations (2.2), the perturbations remain in the domain $x \ll 1$. It will subsequently be clear that, in the general case, this last condition restricts the magnitude of the admissible changes in t. After this change of variables, Hamilton's equations (1.8) retain their form and the factor L^{-1} appears in Eq. (1.10), which makes it convenient to introduce the function $\Psi = L^{-1}\Theta$

$$\frac{dx}{dt} = \frac{\partial\omega}{\partial k}, \quad \frac{dk}{dt} = -\frac{\partial\omega}{\partial x} \tag{2.3}$$

$$\frac{d\Psi}{dt} = \omega(k, x) - k\frac{dx}{dt}, \quad \Psi = \omega t - \int_0^x k(\omega, \xi)d\xi \tag{2.4}$$

The second equality of (2.4) is the integrated first equality, taking into account the fact that $\omega = \mathrm{const}$ on the integral curves of system (2.3). In the principal approximation with respect to L^{-1}, the quantity $\exp[\mathrm{Im}(\Psi L)]$ is the amplification factor of the perturbations. According to the first equation of (2.4), $d\Psi/dt$ is the frequency, calculated in a system of coordinates moving at a velocity dx/dt which clarifies the physical meaning of the quantity Ψ.

We will now investigate the effect of background inhomogeneity on the behaviour of the perturbations, with the assumptions made above. The treatment of perturbations that are close to perturbations on a homogeneous background assumes that $x \ll 1$ and $\tilde{k} = k - k_\upsilon \ll k_\upsilon$, where k_υ is the value of k at the saddle point corresponding to a homogeneous background when $\omega(k) = \omega(k, 0)$. This enables us to use an expansion of the dispersion equation in terms of small x and $\tilde{k} = k - k_\upsilon$, retaining the leading terms:

$$\omega(k, x) = \omega_0 + \upsilon\tilde{k} + \frac{a\tilde{k}^2}{2} + bx; \quad \omega_0 = \omega(k_\upsilon, 0), \quad \upsilon = \frac{\partial\omega}{\partial k}, \quad a = \frac{\partial^2\omega}{\partial k^2}, \quad b = \frac{\partial\omega}{\partial x} \tag{2.5}$$

All derivatives of ω are taken when $x = 0$, $k = k_\upsilon$, where k_υ is a root of Eq. (1.2). The term $a\tilde{k}^2/2$ is retained in expansion (2.5) since υ can take any values, including small ones.

Hamilton's equation (2.3) for the wave number gives

$$\frac{d\tilde{k}}{dt} = -b, \quad \tilde{k} = \tilde{k}_0 - bt \tag{2.6}$$

Here, $\tilde{k}_0$ is the value of $\tilde{k}$ when $t=0$. The fact that $\omega(k, x)$ is a first integral enables us to rewrite Eq. (2.5) in the form

$$\upsilon(\tilde{k}-\tilde{k}_0)+a(\tilde{k}^2-\tilde{k}_0^2)/2+bx = 0 \tag{2.7}$$

Substituting $t=t_1, x=x_1, \tilde{k}=\tilde{k}_0-bt_1, \upsilon=x_1/t_1$ and solving Eq. (2.7) for $\tilde{k}_0$, we obtain

$$\tilde{k}_0 = bt_1/2 \tag{2.8}$$

After substituting the expressions υ, $\tilde{k}$ and $\tilde{k}_0$ into Eqs. (2.6) and (2.5), we obtain (with the accepted accuracy)

$$\tilde{k}(t) = b\left(\frac{t_1}{2}-t\right), \quad x(t) = \frac{x_1}{t_1}t+\frac{abt}{2}(t_1-t) \tag{2.9}$$

We now find the corrections to the quantities Ψ and k_υ when $t=t_1$ due to the inhomogeneity of the background

$$\Delta\Psi = \Delta\omega t_1-\int_0^{x_1}\tilde{k}dx = \frac{bx_1t_1}{2}+\frac{ab^2t_1^3}{24}, \quad \tilde{k} = -\frac{bt_1}{2} \tag{2.10}$$

Here $\Delta\omega$, means the change in ω, calculated in terms of k_0: $\Delta\omega=\upsilon\tilde{k}_0+a\tilde{k}_0^2/2$.
The quantities Ψ and k themselves, when $t=t_1$, have the form

$$\Psi = \omega_0(k_\upsilon)t_1-k_\upsilon x_1+\frac{bx_1t_1}{2}+\frac{ab^2t_1^3}{24}, \quad k = k_\upsilon-\frac{bt_1}{2} \tag{2.11}$$

Equalities (2.11) enable us to describe what an observer, located at a given point x_1, "sees" when $t=t_1$. On account of the background inhomogeneity, the complex amplification factor of the perturbations $\exp(-i\Psi L)$ acquires the additional factor $\exp(-i\Delta\Psi L)$. The quantity $\tilde{k}$ is added to the value $k=k_\upsilon$, calculated ignoring the background inhomogeneity, and the quantities $\Delta\Psi$ and $\tilde{k}$ are defined by equalities (2.10).

According to the first equality of (2.11), the quantity Ψ involves linear terms in x_1 and t_1, corresponding to the evolution of perturbations on the homogeneous background, and quadratic and cubic terms due to the background inhomogeneity. It is well known that the domain of growth of perturbations on a homogeneous background is represented in the x, t plane in the form of sectors.

We shall assume that the linear terms single out a sector in the x, t plane with boundaries $x=\upsilon_1t$ and $x=\upsilon_2t$ in which $\mathrm{Im}\Psi_0>0$, where Ψ_0 is the linear part of expression (2.11). We will now examine how the two remaining terms, arising due to background inhomogeneity, affect the position of the domain of growing perturbations $\Psi>0$. The quadratic term in expressions (2.11) in the neighbourhood of each of the boundaries can be rewritten in the form $b\upsilon_it_1^2/2$. If υ_1 and υ_2 have the same sign, then, depending on the sign of Imb, the effect of this term leads to an enlargement or a contraction of the domain $\Psi>0$ on both sides, compared with the sector $\Psi_0>0$. If υ_1 and υ_2 have opposite signs, the boundaries of the domain $\Psi>0$ will be displaced to one side. The effect of the cubic terms always leads to an enlargement or contraction of the instability zone on both sides. Since $\partial\mathrm{Im}(\Psi_0)/\partial x$ on each ray $x=\upsilon t$ is independent of t, the deviation of the boundaries of the domain $\psi>0$ from the boundaries of the domain $\Psi_0>0$ is determined, spark from a constant factor, by the magnitude of the last two terms in expression (2.11). Account must be taken of the fact that $\partial\Psi_0/\partial x$ has opposite signs on the rays $x=\upsilon_1t$ and $x=\upsilon_2t$.

The trajectory of the perturbation $x(t)$, represented by the second equality of (2.9), is a parabola in the complex plane x. The shape of the trajectory is determined by the fact that the acceleration d^2x/dt^2 of the point $x(t)$ is constant and equal to $-ab$, that is, the motion of the point $x(t)$ is identical to the motion of a particle in a homogeneous force field which is inclined to the Re(x) axis at an angle arg($-ab$). An initial velocity, in equality (2.9), which is equal to $x_1/t_1+abt_1/2$ is selected in such a way that $x(t_1)=x_1$.

If $x_1/t_1\gg|ab|t_1/2$, the trajectory is close to the segment $[0, x_1]$ of the real x axis. Otherwise, the trajectory is a segment of a parabola, which is differs essentially from the above mentioned segment of the Re(x) axis. At the same time, it may turn out that a branch point of the function $k(\omega, x)$ for a value of ω corresponding to the trajectory considered lies between the real x axis and the trajectory. A branch point, as can easily be calculated using dispersion relation (2.5), is located at $x=x_*$, where

$$x_* = \frac{x_1^2}{2abt_1^2}+\frac{abt_1^2}{8}+\frac{x_1}{2}$$

and is the focus of the parabola representing the trajectory of the propagation of the perturbations. If $x_1/t_1^2<|ab|/2$, it lies between the real x axis and the trajectory. If $x_1=|ab|t_1^2/2$, the branch point lies on the real x axis $\mathrm{Im}(x_*)=0$, $\mathrm{Re}(x_*)=x_1(1+\cos\beta)/2$, where $\beta=\arg(ab)$.

The position of the branch point of the function $k(\omega, x)$ enables us, when finding Ψ using the last equality of (2.4), to take advantage of other integration paths different from the trajectories defined by Hamilton's equations. For instance, when $|x_1|/t_1^2>|ab|/2$, we can integrate along the real x axis. In the case of the opposite sign, the integration path should go around the turning point, leaving the real x axis.

Note that, if $\mathrm{Im}(ab)=0$, then, when $t_1^2\gg x_1/(ab)$, the point x_* lies on the real axis outside the segment $[0,x_1]$ and the trajectory $x(t)$, defined by equality (2.9), goes from $x=0$ to x_* and, then, changing the value of k, goes back, arriving at the point x_1 when $t=t_1$. This situation corresponds to the reflection of the perturbation from the turning point $x=x_*$. When $\mathrm{Im}(ab)\neq 0$, passage around the branch point $x=x_*$, which is accompanied by changing the branch of the wave number, can also be treated as a reflection of a perturbation from a background inhomogeneity.

3. Refined description of the solution

We shall now consider an expansion of the dispersion equation which takes account of additional terms in the expansion of $\omega(\tilde{k}, x)$ (the last two terms)

$$\omega(k, x) = \omega_0 + \upsilon\tilde{k} + a\tilde{k}^2/2 + bx + c\tilde{k}x + gx^2/2 \tag{3.1}$$

Hamilton's equations have the form

$$\frac{dx}{dt} = \upsilon + a\tilde{k} + cx, \quad \frac{dk}{dt} = -(b + c\tilde{k} + gx) \tag{3.2}$$

We shall, as previously, seek solutions of Hamilton's equations by expanding the functions $x(t)$ and $\tilde{k}(t)$ in series in the time t and retaining small terms not higher than the second order of magnitude.

The coefficients of the expansion of the functions $x(t)$ and $k(t)$ are found from Hamilton's equations and, at the same time, the second derivatives are found by differentiation of these equations and are calculated when $x = 0$, $\tilde{k} = \tilde{k}_0$. With the accepted accuracy, we obtain

$$x(t) = (\upsilon + a\tilde{k}_0)t + (c\upsilon - ab)t^2/2, \quad \tilde{k}(t) = \tilde{k}_0 - (b + c\tilde{k}_0)t + (cb - g\upsilon)t^2/2 \tag{3.3}$$

The initial value $\tilde{k}_0$ is found from the condition for the trajectory of the perturbation to arive at the point of observation $x = x_1$, $t = t_1$. As a result, we obtain

$$\tilde{k}_0 = b_1 t_1/2 - cx_1/2a$$

The first term in this expression is coincides with the value of $\tilde{k}_0$ in the preceding case when inhomogeneity was only taken into account by the term bx.

The quantity $\Delta\Psi$, which is the correction to the quantity Ψ at the point of observation, is expressed, with the accepted accuracy, as follows:

$$\Delta\Psi = \frac{bx_1t_1}{2} + \frac{ab^2t_1^3}{24} - \frac{bcx_1t_1^2}{12} - \frac{c^2t_1x_1^2}{8a} + \frac{gx_1^2t_1}{6} \tag{3.4}$$

and, consequently,

$$\Psi = \omega_0(k_\upsilon)t_1 - k_\upsilon x_1 + \Delta\Psi \tag{3.5}$$

As in the preceding case, we can find the turning points, from the condition for the propagation velocity of the perturbations to varish and the condition for them to reach the real axis. We mention that, unlike in the preceding case, there are two turning points, one of which is close to the point found earlier while the other lies in a remote region.

Note that the solution obtained earlier, which is represented by equalities (2.9) and (2.10), is an exact solution of the Hamilton and Hamilton–Jacobi equations (2.3) and (2.4) in the case of the dispersion equation (2.5). This solution is applicable for arbitrary values of t if equality (2.5) is the exact dispersion equation with υ = const, rather than the approximate representation of it in the form of an expansion in $\tilde{k}$ and x. In contrast to this, the solution (3.3)–(3.5) is approximate and holds for not too long time intervals. This latter fact is unimportant if expression (3.1) is considered to be an approximate one. It is easy to obtain the corresponding exact solution of Eqs. (3.2) which are suitable for any time interval. This solution is presented below when considering an example.

4. Example. Comparison of the exact and approximate solutions

As an illustrative example, we shall consider a system characterized by the dispersion equation

$$\Omega = i(AK - BK^2) + CKX \tag{4.1}$$

The corresponding Hamilton's equations have previously been integrated exactly.[11] We will now compare the approximate solutions obtained with the above-mentioned exact solution of Hamilton's equations and estimate the error due to the approximate solution of these equations.

Equation (4.1) without the last term corresponds to the case of linear perturbations of a combustion front when the heat released during combustion is not very large.[12] If the tangential component of the velocity at the flame front in the unperturbed flow depends linearly on the x coordinate, the dispersion equation can be approximately written in the form of (4.1).[11] In Eq. (4.1), A is a constant characterizing the instability, subject to the condition that the propagation velocity of the flame front through the hot mixture is constant,[13] B is a constant, which relates the propagation velocity of a distorted front through a hot mixture to its curvature,[14] and C is a constant which characterizes the dependence of the tangential component of the velocity on the coordinate. A, B and C are positive quantities. Using a change in the scales of ω and k, this equation can be reduced to a form which has been used earlier:[11]

$$\omega(k, x) = i(k - k^2) + kx \tag{4.2}$$

For this reduction, instead of the variables k, x, ω and t corresponding to Eq, (4.1), we introduce new variables, corresponding to Eq. (4.2), according to the equalities

$$k = \frac{BK}{A}, \quad x = \frac{CX}{A}, \quad \omega = \frac{B\Omega}{A^2}, \quad t = \frac{A^2\tau}{C}$$

Dispersion equation (4.2) can be rewritten in the form

$$\omega = \frac{i(1+\upsilon^2)}{4} + \upsilon\tilde{k} - i\tilde{k}^2 + \frac{1+i\upsilon}{2}x + \tilde{k}x, \quad \tilde{k} = k - k_\upsilon, \quad k_\upsilon = \frac{1+i\upsilon}{2} \tag{4.3}$$

We will now briefly present results obtained earlier.[11] It is well known that the Hamiltonian $\omega(k,x)$ is the integral of Hamilton's equations which enables us to write Hamilton's equations corresponding to equality (4.2) in the form

$$\frac{dk}{dt} = -k, \quad x = \frac{\omega}{k} - i + ik \tag{4.4}$$

This system is integrated in the explicit form:

$$k = k_0e^{-t}, \quad k_0 = \frac{e^{t_1}-1}{e^{t_1}-e^{-t_1}} + i\frac{x_1 - x_0e^{t_1}}{e^{t_1}-e^{-t_1}}, \quad x = \frac{\omega}{k_0}e^t - i + ik_0e^{-t}$$

$$\Psi = \omega t_1 - \int\limits_{x_0}^{x_1} k_0e^{-t}\frac{dx}{dt}dt = \frac{i}{2}(1-e^{-2t_1})k_0^2 \tag{4.5}$$

$$\operatorname{Im}\Psi = \frac{1}{2}(1-e^{-2t_1})\operatorname{Re}k_0^2, \quad \operatorname{Im}\Psi = (1-(x_1e^{-t_1}-x_0)^2)/2 \text{ as } t_1\to\infty \tag{4.6}$$

The initial data when $t = 0$ are assumed to be concentrated in the neighbourhood of the point $x = x_0$, and x_1, t_1 is the point of observation. The amplification factor was found to be bounded everywhere. On the curves $x_1 = (x_0 + q)e^{t_1} - q$ (q = const), when $t_1 \to \infty$, it takes the value

$$\lim_{t\to\infty} \operatorname{Im}\Psi = \frac{1}{2}(1-q^2)$$

The domain, where this limiting amplification factor is positive, lies between the curves

$$x(t_1) = (x_0-1)e^{t_1} + 1 \text{ and } x(t_1) = (x_0+1)e^{t_1} - 1.$$

It reaches a maximum equal to 1/2 on the curve $x(t_1) = x_0e^{t_1}$.

Note that, in the homogeneous case (when we put $x = x_0$ = const in equality (4.2)) in the x, t plane, the perturbations grow on the straight lines $x = x_0 + \upsilon$, υ = const when $x_0 - \sqrt{2} < \upsilon < x_0 + \sqrt{2}$ and, moreover, the amplification factor increases linearly with time. In the inhomogeneous case, the perturbation growth is found to be bounded everywhere due to the term kx and the domain occupied by non-decaying perturbations expands exponentially. It can be seen from Eqs (4.4) that all the wave numbers decrease as exp(-t), precisely because of the presence of the term kx in the dispersion equation which, eventually, leads to stopping the growth of the quantity ImΨ and to decreasing the curvature of the front as exp(-2t). If we return to the dimension time τ, the curvature of the combustion front will decrease as exp(-2$A\tau/C$).

Using the general formulae obtained earlier, we can find for both approximate methods of taking into account the inhomogeneity in the background the quantities characterizing the perturbations, that is, the wave number, trajectory and propagation velocity, turning points as well as the amplification factor, which is equal to Im(ΨL), and the boundaries of the domain in the x, t plane where Im(Ψ) > 0. Note that, in the case of the dispersion equation considered, the approximate solution coincides with the expansion in time of the exact solution.

The boundaries of the domains of perturbations growth, corresponding to the exact solution (the thin solid curve), to the solutions of the approximate dispersion equations (2.5) and (3.1) (the dot–dash curves and the dashed curves respectively) and to the solution obtained ignoring the inhomogeneity in the background (the heavy continuous line), are presented in Fig. 1 for one and the same point of observation and the initial point $x_0 = 0$.In view of the symmetry with respect to x_1, only the domain $x_1 \geq 0$ is shown. It can be seen that, for time values $t = 0.8$, the approximate solutions approximate the exact solutions quite well. This corresponds to a domain of perturbation growth, which is arranged, when $-1 < x_1 < 1$, so that the approximate solution is close to the exact solution in the domain where the change in the background becomes significant.

The amplification factors ImΨ are presented in Fig. 2 as a function of x_1 for two instants of time: $t_1 = 0.5$ (the lower group of curves) and $t_1 = 0.9$ (the upper group of curves).

In view of the symmetry, only the domain $x_1 \geq 0$ is shown. The amplification factors are represented by the solid thick curves in the case of the exact solution, by dot-dash and dashed curves in the case of approximate dispersion equations of the form of (2.5) and (3.1) respectively and by thin solid curves in the case of a homogeneous background. The curves representing the amplification factors calculated using a formula of the type of (3.5), graphically coincide with the curves representing the exact solutions. As in Fig. 1, in view of the symmetry with respect to x, only half the graph is shown. Hence, the use of an expansion of the dispersion equation, which differs from a representation

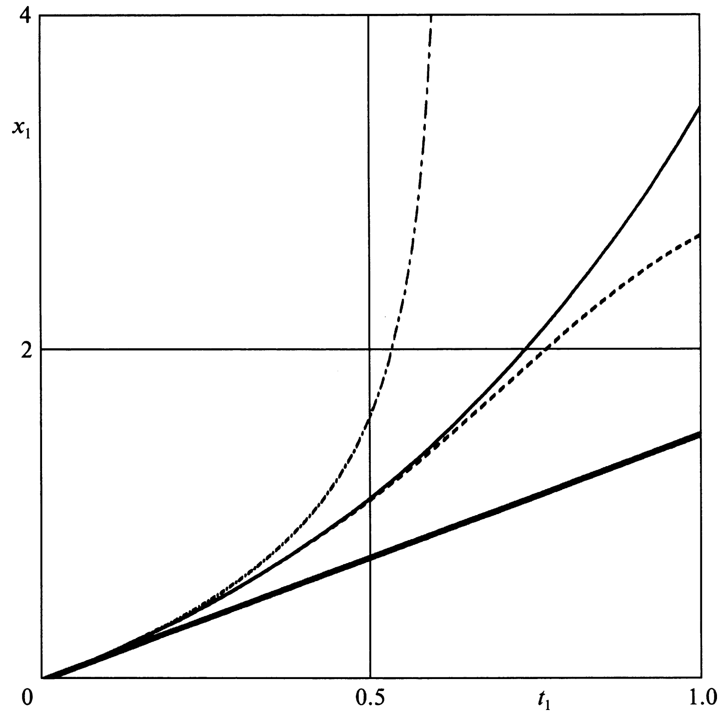

Fig. 1.

of the form of (2.5) sololy in the term $\tilde{k}x$ gives good agreement with the results of the exact solution for the above-mentioned instants of time.

The point $x_0 = 0$, $t_0 = 0$ was taken as the initial point in all cases. The choice of another point x_0 as the initial point leads to a drift of the perturbations along the x axis and to a change in the boundaries of the domains of perturbations growth, which is determined by terms involving $x_0 e^{t_1}$.

The example considered therefore shows that the approximate power series expansion agrees quite well with the exact solution even in the domain where there is an essential change in the background.

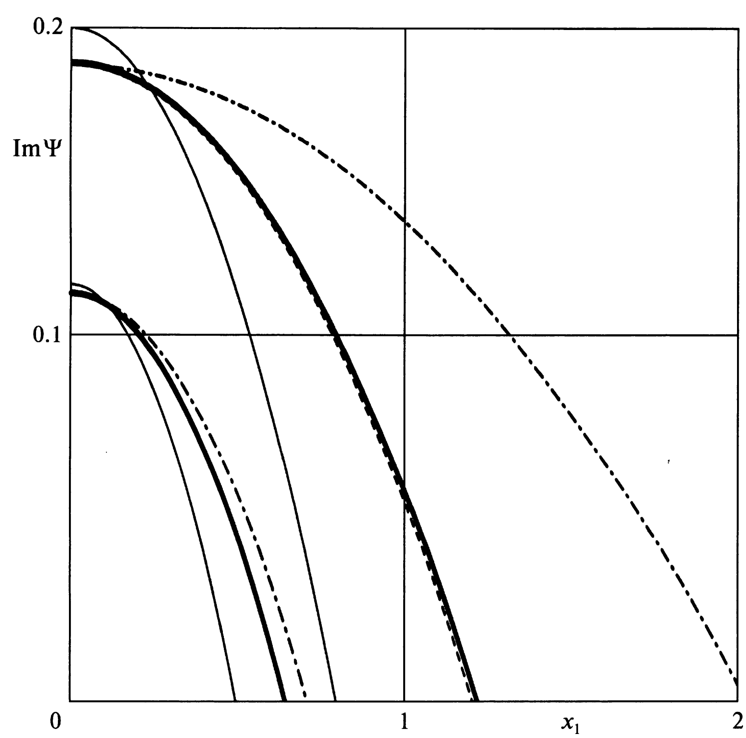

Fig. 2.

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References

1. Whitham GB. *Linear and Nonlinear Waves*. New York: Wiley; 1974.
2. Lifshitz EM, Pitaevskii LP. *Physical Kinetics*. Oxford: Pergamon; 1981.
3. Fedoryuk MF. *The Method of Steepest Descent*. Moscow: Nauka; 1977.
4. Maslov VP. *Operator Methods*. Moscow: Nauka; 1973.
5. Iordanskii SV. The stability of inhomogeneous states and continual integrals. *Zh Exper Teor Fiz* 1988;**94**(7):180–9.
6. Belov VV, Dobrokhotov OYu, Dobrokhotov SYu. Isotropic tori, complex germ and Maslov index, the norm of a form and quasimodes of multidimensional problems. *Mat Zamet* 2001;**69**(4):483–514.
7. Maslov VP. *Asymptotic Methods for Solving Pseudodifferential Equations*. Moscow: Nauka; 1987.
8. Maslov VP. *Perturbation Theory and Asymptotic Methods*. Paris: Dunod; 1972.
9. Wienstein A. The Maslov Gerbe. arXiv:math/0312274v1 [math.SG] 13 Dec. 2003.
10. Hunt RE, Crighton DG. Instability of flows in spatially developing media. *Proc Roy Soc London Ser A* 1991;**435**(1893):109–28.
11. Kulikovskii AG, Lozovskii AV, Pashchenko NT. The evolution of perturbations on a weakly inhomogeneous background. *Prikl Mat Mekh* 2007;**71**(5):761–74.
12. Sivashinsky GI. Nonlinear analysis of hydrodynamic instability in laminr flames. I. Derivation of basic equations. *Acta Astronaut* 1997;**4**(11/12):1177–206.
13. Landau LD. On the theory of slow combustion. *Zh Eksper Teor Fiz* 1944;**14**(6):240–5.
14. Markshtein GH. Experimental and theoretical studies of flame front stability. *J Aeronaut Sci* 1951;**18**(3):199–209.

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